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Nonlinear Stability of Strong Planar Rarefaction Waves for the Relaxation Approximation of Conservation Laws in Several Space Dimensions

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T. P. Liu (1987, *Comm. Math. Phys.* **108**, 153–175) and T. Luo (1997, *J. Differential Equations* **133**, 255–279), our main novelty lies in the fact that the planar rarefaction waves do not need to be small, and in the one-dimensional case, the initial disturbance can also be chosen arbitrarily large. © 2000 Academic Press

1. INTRODUCTION AND THE STATEMENT OF OUR MAIN RESULTS

A system of hyperbolic conservation laws in several space dimensions takes the form

$$\mathbf{u}_t + \sum_{j=1}^{N} \mathbf{F}_j(\mathbf{u})_{x_j} = 0, \qquad \mathbf{u} \in \mathbf{R}^n,\ (t, \mathbf{x}) \in \mathbf{R}^+ \times \mathbf{R}^N. \tag{1.1}$$

To approximate this system from numerical point of view, S. Jin and Z. P. Xin[10] proposed the relaxation system

$$\begin{cases} \mathbf{u}_t + \sum_{j=1}^{N} (\mathbf{v}_j)_{x_j} = 0, & \mathbf{u} \in \mathbf{R}^n,\ \mathbf{v}_j \in \mathbf{R}^n, \\ (\mathbf{v}_j)_t + \mathbf{A}_j \mathbf{u}_{x_j} = -\dfrac{\mathbf{v}_j - \mathbf{F}_j(\mathbf{u})}{\varepsilon}, & j = 1, 2, ..., N, \end{cases} \tag{1.2}$$

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where $\mathbf{A}_j = a_j \mathbf{I}$, $a_j > 0$ $(j = 1, 2, ..., N)$, and $\mathbf{I}$ is the $n \times n$ identity matrix. A numerical scheme to solve (1.2) is designed in [10] which yields satisfactory numerical solutions to hyperbolic conservation laws (1.1). The main features of this scheme are its generality and simplicity.

In this paper, as in [13, 15], we consider the nonlinear time asymptotic stability of strong planar rarefaction waves for the above mentioned relaxation approximation to the corresponding scalar conservation laws in several space dimensions. Without loss of generality, we will be concerned with the scalar conservation laws in two dimensions

$$u_t + f(u)_x + g(u)_y = 0, \qquad u \in \mathbf{R}. \tag{1.3}$$

Its relaxation approximation proposed by S. Jin and Z. P. Xin in [10] becomes

$$\begin{cases} u_t + v_{1x} + v_{2y} = 0, \\ v_{1t} + a_1 u_x = -\dfrac{v_1 - f(u)}{\varepsilon}, \\ v_{2t} + a_2 u_y = -\dfrac{v_2 - g(u)}{\varepsilon}. \end{cases} \qquad u, v_1, v_2 \in \mathbf{R}, \tag{1.4}$$

Here a_1 and a_2 are two positive constants satisfying the subcharacteristic condition

$$\sup_{u \in M} \left\{ \frac{[f'(u)]^2}{a_1} + \frac{[g'(u)]^2}{a_2} \right\} < 1, \tag{$(SC)_2$}$$

where $M \subset \mathbf{R}$ is the state space whose precise definition will be specified later.

We are interested in the Cauchy problem to the system (1.4) with initial conditions

$$\begin{cases} u(0, x, y) = u_0(x, y), \\ v_1(0, x, y) = v_{10}(x, y), \\ v_2(0, x, y) = v_{20}(x, y), \end{cases} \tag{1.5}$$

which satisfy

$$\begin{cases} \lim\limits_{x \to \pm\infty} \|u_0(x, y) - u^{\pm}\|_{L^\infty(\mathbf{R}_y)} = 0, \\ \lim\limits_{x \to \pm\infty} \|v_{10}(x, y) - f(u^{\pm})\|_{L^\infty(\mathbf{R}_y)} = 0, \\ \lim\limits_{x \to \pm\infty} \|v_{20}(x, y) - g(u^{\pm})\|_{L^\infty(\mathbf{R}_y)} = 0. \end{cases} \tag{1.6}$$

Here u^- and u^+ are two constants satisfying $u^- < u^+$.

Furthermore, we assume that the flux functions $f(u)$ and $g(u)$ are smooth enough and that the equation (1.3) is genuinely nonlinear in x-direction, i.e., there exists a positive constant γ such that

$$\inf_{u \in M \cup [-B(N_0), B(N_0)]} f''(u) \geqslant \gamma. \tag{1.7}$$

(For the precise definitions of M and $B(N_0)$, see Theorem 1.1 and Theorem 1.2 below.)

For such a flux function $f(u)$ and the two constants u^- and u^+, a planar rarefaction wave is the entropic weak solution $r(t, x)$ of the following Cauchy problem

$$\begin{cases} r_t + f(r)_x = 0, \\ r(0, x) = r_0(x), \end{cases} \tag{1.8}$$

with $r_0(x)$ given by

$$r_0(x) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0. \end{cases} \tag{1.9}$$

It is well known that $r(t, x)$ can be given explicitly by

$$r(t, x) = \begin{cases} u^-, & \dfrac{x}{t} < f'(u^-), \\ (f')^{-1}\left(\dfrac{x}{t}\right), & f'(u^-) \leqslant \dfrac{x}{t} \leqslant f'(u^+), \\ u^+, & \dfrac{x}{t} > f'(u^+). \end{cases} \tag{1.10}$$

Under the above notations, the main purpose of this paper is to study the global solvability of the Cauchy problem (1.4)–(1.5), and to compare $(u, v_1, v_2)(t, x, y)$ with $(r, f(r), g(r))(t, x)$ as the time t tends to infinity. Such a problem was originally considered by T. P. Liu in [13] in the one-dimensional case, yet for the general 2×2 hyperbolic systems of conservation laws with relaxation, and later by T. Luo in [15] for the Cauchy problem (1.4)–(1.5). However, in these two papers the planar rarefaction waves are assumed to be weak, and the initial disturbance is small in some Sobolev space. Our main results in this paper contain two parts: First, for the one-dimensional case, i.e., for the following Cauchy problem

$$\begin{cases} u_t + v_{1x} = 0, \\ v_{1t} + a_1 u_x = -\dfrac{v_1 - f(u)}{\varepsilon}, \\ (u(0, x), v_1(0, x)) = (u_0(x), v_{10}(x)), \end{cases} \tag{1.11}$$

under the subcharacteristic condition

$$\sup_{|u| \leqslant B(N_0)} |f'(u)| < \sqrt{a_1}, \tag{SC$_1$}$$

where

$$\begin{cases} N_0 = \max\{\|u_0(x)\|_{L^\infty(\mathbf{R})}, \|v_{10}(x)\|_{L^\infty(\mathbf{R})}\}, \\ F(N_0) = \sup\limits_{|u| \leqslant N_0} |f'(u)|, \\ B(N_0) = 2N_0 + F(2N_0), \end{cases} \tag{1.12}$$

we can get the same nonlinear stability results as those obtained by T. P. Liu in [13] without the smallness conditions on $|u^+ - u^-|$ and the initial disturbance.

Second, for the Cauchy problem (1.4)-(1.5), we can obtain the same stability results as those obtained by T. Luo in [15] without the smallness condition on $|u^+ - u^-|$. To state our results precisely, following A. Matsumura and K. Nishihara[20], we introduce the smooth rarefaction wave, which is the smooth solution $\phi(t, x)$ of the Cauchy problem

$$\begin{cases} \phi_t + f(\phi)_x = 0, \\ \phi(0, x) = \phi_0(x). \end{cases} \tag{1.13}$$

Here $\phi_0(x)$ is defined by

$$\phi_0(x) = \frac{u^+ + u^-}{2} + \frac{u^+ - u^-}{2} \kappa \int_0^x \frac{dz}{1+z^2},$$

and

$$\kappa = \left(\int_0^\infty \frac{dz}{1+z^2} \right)^{-1}.$$

Now we are in position to state our first result on the Cauchy problem (1.11).

THEOREM 1.1 (The One-Dimensional Case). *Suppose that* $(u_0(x) - \phi_0(x), v_{10}(x) - f(\phi_0(x))) \in H^2(\mathbf{R})$ *and satisfies the subcharacteristic condition* (SC)$_1$, *then the Cauchy problem* (1.11) *admits a unique global smooth solution* $(u(t, x), v_1(t, x))$ *which satisfies*

$$\sup_{t \geqslant 0} \|u(t, x)\|_{L^\infty(\mathbf{R})} \leqslant B(N_0) \tag{1.14}$$

(*consequently* $u(t, x)$ *satisfies* $(\mathrm{SC})_1$ *for all* $t>0$, $x\in\mathbf{R}$) *and*

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\left\{\left|\frac{\partial^i}{\partial x^i}(u(t,x)-\phi(t,x), v_1(t,x)-f(\phi(t,x)))\right|\right\}=0, \qquad i=1,2, \tag{1.15}$$

or equivalently

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\{|(u(t,x)-r(t,x), v_1(t,x)-f(r(t,x)))|\}=0. \tag{1.16}$$

Based on the one-dimensional stability result, we can get the following stability result for the multidimensional case

THEOREM 1.2 (The Multidimensional Case). *Suppose that* $(u_0(x,y)-\phi_0(x), v_{10}(x,y)-f(\phi_0(x)), v_{20}(x,y)-g(\phi_0(x)))\in H^4(\mathbf{R}^2)$ *and let*

$$\alpha=\|u_0(x,y)-\phi_0(x)\|_{H^4(\mathbf{R}^2)}+\left\|\frac{\partial}{\partial x}(v_{10}(x,y)-f(\phi_0(x)))\right\|_{H^3(\mathbf{R}^2)} + \left\|\frac{\partial}{\partial y}v_{20}(x,y)\right\|_{H^3(\mathbf{R}^2)}.$$

Then, for each given constant $\varepsilon_0>0$, *there exists a constant* $\alpha_0>0$ *such that if* $\alpha\leqslant\alpha_0$ *and the subcharacteristic condition* $(\mathrm{SC})_2$ *is satisfied with*

$$M=[-B(N_1)-\varepsilon_0, B(N_1)+\varepsilon_0], \tag{1.17}$$

where

$$N_1=\max\{|u^-|, |u^+|, \|f(\phi_0(x))\|_{L^\infty(\mathbf{R})}\},$$

the Cauchy problem (1.4)–(1.5) *has a unique global smooth solution* $(u, v_1, v_2)(t,x,y)$ *which satisfies*

$$\|u(t,x,y)\|_{L^\infty(\mathbf{R}^2)}\leqslant B(N_1)+\varepsilon_0, \tag{1.18}$$

and

$$\begin{cases}\lim_{t\to+\infty}\|u(t,x,y)-\phi(t,x)\|_{L^\infty(\mathbf{R}^2)}=0,\\ \lim_{t\to+\infty}\|v_1(t,x,y)-f(\phi(t,x))\|_{L^\infty(\mathbf{R}^2)}=0,\\ \lim_{t\to+\infty}\|v_2(t,x,y)-g(\phi(t,x))\|_{L^\infty(\mathbf{R}^2)}=0.\end{cases} \tag{1.19}$$

Consequently

$$\begin{cases} \lim\limits_{t\to+\infty} \|u(t,x,y)-r(t,x)\|_{L^\infty(\mathbf{R}^2)}=0, \\ \lim\limits_{t\to+\infty} \|v_1(t,x,y)-f(r(t,x))\|_{L^\infty(\mathbf{R}^2)}=0, \\ \lim\limits_{t\to+\infty} \|v_2(t,x,y)-g(r(t,x))\|_{L^\infty(\mathbf{R}^2)}=0. \end{cases} \tag{1.20}$$

Remark 1.1. Our method in proving Theorem 1.2 applies to higher space dimensional case.

Remark 1.2. For the Cauchy problem (1.4)–(1.5), if one can get the following time independent $L^\infty(\mathbf{R}^2)$ *a priori* estimates

$$\|(u(t,x,y), v_1(t,x,y), v_2(t,x,y))\|_{L^\infty(\mathbf{R}^2)} \leqslant C_0 < \infty, \qquad t \geqslant 0, \tag{1.21}$$

then the results of Theorem 1.2 will still hold even for large initial disturbance.

Remark 1.3. As pointed out by T. Luo in [15], the choice of the x-direction in this paper involves no loss of generality, because we can reduce the general situation to this case by suitable change of coordinates.

Remark 1.4. After a rescaling, we only need to study the case $\varepsilon=1$ in the rest of this paper.

Remark 1.5. The smallness assumption we imposed on the initial disturbance is slightly weaker than those proposed by T. Luo in [15]. In our Theorem 1.2 the smallness of ε_0 is not needed anymore. In fact, if we assume that

$$\left|\frac{\partial^i f(u)}{\partial u^i}\right| + \left|\frac{\partial^i g(u)}{\partial u^i}\right| \leqslant O(1)(1+|u|^p), \qquad 0<p<1,\ i=1,2,3,4, \tag{1.22}$$

then checking the proof of Theorem 1.2 carefully, we can see that the constant $C_2(\varepsilon_0)$ in (3.29) satisfies

$$\lim_{\varepsilon_0\to+\infty} \frac{\varepsilon_0}{\sqrt{C_2(\varepsilon_0)}} = +\infty,$$

and in this case, no smallness assumption is needed on the initial disturbance.

Finally, we point out that the relaxation mechanism arises in many physical situations, for example, gases not in thermodynamic equilibrium, kinetic theory, chromatography, river flow, traffic flows, and more general waves, cf. G. Whitham [27]. The general 2×2 hyperbolic systems of conservation laws with relaxation in the form

$$\begin{cases} u_t + f(u, v)_x = 0, \\ v_t + g(u, v)_x = \dfrac{1}{\varepsilon}\, h(u, v), \end{cases} \tag{1.23}$$

was first analyzed by T. P. Liu in [13] to justify some nonlinear stability criteria for diffusion waves, expansion waves, and traveling waves. Since then, the stability of certain elementary waves was studied by H. Liu, C. Woo, and T. Yang [12], T. Luo [15], T. Luo and Z. P. Xin [16], C. Mascia and R. Natalini [18], M. Mei and T. Yang [22], R. H. Pan [26], C. J. Zhu [31], and P. Zingano [32], etc. The problem on the convergence to the diffusion waves was given by I.-L. Chern in [4]. Related results on the relaxation time limit can be found in G. Chen, C. Levermore, and T. P. Liu [2], G. Chen and T. P. Liu [3], C. Lattanzio and P. Marcati [11], R. Natalini [23], etc. For a more complete literature in this direction, we refer the interested reader to the recent book of L. Hsiao [7] and the survey paper of R. Natalini [24].

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 and Section 3 is devoted to proving Theorem 1.2.

Notations. Throughout this paper, we use $O(1)$ to denote a generic positive constant independent of t, and use $\|.\|_s$ to denote the norm in $H^s(\mathbf{R}^2)$ with $\|.\| = \|.\|_0$. The symbol $\iiint$ will stand for the triple integral over $[0, t]\times\mathbf{R}^2$ and $\iint$ will represent the integral on $\mathbf{R}^2$ unless otherwise stated.

2. THE PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. For this purpose, we first give some properties of the smooth rarefaction wave $\phi(t, x)$.

LEMMA 2.1. *$\phi(t, x)$ is a sufficiently smooth function which satisfies*

(i) $u^- < \phi(t, x) < u^+$, $\phi_x(t, x) > 0$ *for* $(t, x)\in[0, \infty)\times\mathbf{R}$;

(ii) *For all* $p\in[1, \infty]$, *there is a constant* $C_{p,\delta}$ *such that*

$$\begin{cases} \|\phi_x(t)\|_{L^p(\mathbf{R})} \leqslant C_{p,\delta}(1+t)^{-1+1/p}, \\ \left\| \dfrac{\partial^i\phi(t)}{\partial x^i} \right\|_{L^p(\mathbf{R})} \leqslant C_{p,\delta}(1+t)^{-(1+i)/2+1/2p}, \qquad i = 2, 3. \end{cases} \tag{2.1}$$

Here $\delta = u^+ - u^-$.

(iii) *For all* $p \in (1, \infty]$, *there is a constant* $C_{p,\delta}$ *such that*

$$\|\phi(t,x) - r(t,x)\|_{L^p(\mathbf{R})} \leqslant C_{p,\delta} t^{-(p-1)/2p} \tag{2.2}$$

for $t > 0$.

The proof of Lemma 2.1 can be found in [9, 19–21] and thus, we omit the details.

For the Cauchy problem (1.11), from Theorem 3.1 of [23], we have the following global existence result

LEMMA 2.2 [23] (Global Existence Result). *Under the conditions of Theorem* 1.1, *the Cauchy problem* (1.11) *admits a unique global smooth solution* $(u(t,x), v_1(t,x))$ *which satisfies*

$$\begin{cases} \|u(t,x)\|_{L^\infty(\mathbf{R})} \leqslant B(N_0), \\ \|v_1(t,x)\|_{L^\infty(\mathbf{R})} \leqslant \sqrt{a_1}\, B(N_0). \end{cases} \tag{2.3}$$

As a direct corollary, for each $t \geqslant 0, x \in \mathbf{R}$, *the global smooth solution* $(u(t,x), v_1(t,x))$ *satisfies the subcharacteristic condition* $(\mathrm{SC})_1$.

Proof. The proof of Lemma 2.2 is essentially due to R. Natalini [23]. The only difference lies in that in Theorem 3.1 of [23], the unique solution $(u(t,x), v_1(t,x))$ obtained here just belongs to $L^\infty([0,\infty); L^\infty(\mathbf{R})^2) \cap C([0,\infty); L^1_{loc}(\mathbf{R})^2)$. But notice that the system $(1.11)_1$–$(1.12)_2$ is semilinear and the initial data lies in a bounded subset of $C^1(\mathbf{R})^2$, we can easily deduce that such a $(u(t,x), v_1(t,x))$ obtained above is indeed a classical solution to the Cauchy problem (1.11). This ends the proof of Lemma 2.2.

Having obtained Lemma 2.2, to complete the proof of Theorem 1.1, we need only to get the temporal decay estimate (1.15) since (1.16) is a direct consequence of (1.15) and (2.2). For this purpose, let

$$\begin{cases} w(t,x) = u(t,x) - \phi(t,x), \\ z(t,x) = v_1(t,x) - f(\phi(t,x)), \end{cases} \tag{2.4}$$

then, it is easy to see that $(w(t,x), z(t,x))$ satisfies the Cauchy problem

$$\begin{cases} w_t + z_x = 0, \\ z_t + a_1 w_x + z = [f(w+\phi) - f(\phi)] - (a_1\phi_x + f(\phi)_t), \\ (w(0,x), z(0,x)) = (w_0(x), z_0(x)) = (u_0(x) - \phi_0(x), v_{10}(x) - f(\phi_0(x))), \end{cases} \tag{2.5}$$

and by employing the arguments used by T. P. Liu in [13], the proof of (1.15) is then reduced to the proof of the following result

LEMMA 2.3 (Energy Estimates). *Under the conditions of Theorem* 1.1, *we have that for each* $0 < t < \infty$, *the unique global smooth solution* $(w(t, x), z(t, x))$ *of the Cauchy problem* (2.5) *satisfies*

$$\|w(t)\|^2_{H^2(\mathbf{R})} + \|w_t(t)\|^2_{H^1(\mathbf{R})} + \|w_{tt}(t)\|^2_{L^2(\mathbf{R})}$$
$$+ \int_0^t (\|\sqrt{\phi_x} w\|^2_{L^2(\mathbf{R})} + \|w_x\|^2_{H^1(\mathbf{R})} + \|w_t\|^2_{H^1(\mathbf{R})} + \|w_{tt}\|^2_{L^2(\mathbf{R})})(s)\, ds$$
$$\leqslant O(1)(\|w_0(x)\|^2_{H^2(\mathbf{R})} + \|z_{0x}(x)\|^2_{H^1(\mathbf{R})} + 1), \tag{2.6}$$

and

$$\|z(t)\|^2_{H^2(\mathbf{R})} + \|z_t(t)\|^2_{H^1(\mathbf{R})} + \|z_{tt}(t)\|^2_{L^2(\mathbf{R})}$$
$$+ \int_0^t (\|z_x\|^2_{H^1(\mathbf{R})} + \|z_t\|^2_{H^1(\mathbf{R})} + \|z_{tt}\|^2_{L^2(\mathbf{R})})(s)\, ds$$
$$\leqslant O(1)(\|w_0(x)\|^2_{H^2(\mathbf{R})} + \|z_0(x)\|^2_{H^2(\mathbf{R})} + 1). \tag{2.7}$$

We only prove (2.6) since (2.7) is a direct corollary of (2.6), $(2.5)_1$, and $(2.5)_2$. To this end, we first notice from $(2.5)_1$ and $(2.5)_2$ that $w(t, x)$ satisfies

$$\begin{cases} w_{tt} + w_t - a_1 w_{xx} = -[f(w+\phi) - f(\phi)]_x + (a_1 \phi_{xx} + f(\phi)_{xt}), \\ w(0, x) = w_0(x), \\ w_t(0, x) = z_{0x}(x), \end{cases} \tag{2.8}$$

and for the basic energy estimate for $w(t, x)$, we can get

LEMMA 2.4 (Basic Energy Estimate). *Under the assumptions of Theorem* 1.1, *we have*

$$\int_{\mathbf{R}} (w^2 + w_x^2 + w_t^2)(t)\, dx + \int_0^t \int_{\mathbf{R}} (\phi_x w^2 + w_x^2 + w_t^2)(s, x)\, dx\, ds$$
$$\leqslant O(1)(\|w_0(x)\|^2_{H^1(\mathbf{R})} + \|z_{0x}(x)\|^2_{L^2(\mathbf{R})} + 1). \tag{2.9}$$

Proof. Multiplying $(2.8)_1$ by $\frac{1}{d} w + w_t$ and integrating the results over $[0, t] \times \mathbf{R}$, after some integrations by parts, we can get

$$\frac{1}{2d} \int_{\mathbf{R}} w^2\, dx + \frac{1}{2} \int_{\mathbf{R}} w_t^2\, dx + \frac{a_1}{2} \int_{\mathbf{R}} w_x^2\, dx$$
$$+ \frac{a_1}{d} \int_0^t \int_{\mathbf{R}} w_x^2\, dx\, ds + \left(1 - \frac{1}{d}\right) \int_0^t \int_{\mathbf{R}} w_t^2\, dx\, ds$$

$$
\begin{aligned}
&= O(1)(\|w_0(x)\|^2_{H^1(\mathbf{R})} + \|z_{0x}(x)\|^2_{L^2(\mathbf{R})}) - \frac{1}{\mathrm{d}}\int_{\mathbf{R}} ww_t\, dx \\
&\quad - \frac{1}{d}\int_0^t \int_{\mathbf{R}} w[f(w+\phi) - f(\phi)]_x\, dx\, ds \\
&\quad - \int_0^t \int_{\mathbf{R}} w_t[f(w+\phi) - f(\phi)]_x\, dx\, ds \\
&\quad + O(1)\int_0^t \int_{\mathbf{R}} (|w| + |w_t|)(|\phi_{xx}| + |\phi_x|^2)\, dx\, ds \\
&= \sum_{i=1}^{5} I_i.
\end{aligned}
\tag{2.10}
$$

By employing the Cauchy–Schwarz inequality, we have

$$
I_1 \leqslant \frac{1}{4d}\int_{\mathbf{R}} w^2\, dx + \frac{1}{d}\int_{\mathbf{R}} w_t^2\, dx,
\tag{2.11}
$$

$$
\begin{aligned}
I_5 &\leqslant O(1)\int_0^t (\|\phi_{xx}(s)\|_{L^2(\mathbf{R})} + \|\phi_x(s)\|^2_{L^4(\mathbf{R})})\, \|w(s)\|_{L^2(\mathbf{R})}\, ds \\
&\quad + \varepsilon \int_0^t \int_{\mathbf{R}} w_t^2\, dx\, ds \\
&\quad + O(1)\int_0^t \int_{\mathbf{R}} (|\phi_{xx}|^2 + |\phi_x|^4)(s, x)\, dx\, ds \\
&\leqslant O(1)\left(1 + \int_0^t (1+t)^{-5/4}\|w(s)\|^2_{L^2(\mathbf{R})}\, ds\right) \\
&\quad + \varepsilon \int_0^t \int_{\mathbf{R}} w_t^2\, dx\, ds,
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
I_3 &= \frac{1}{d}\int_0^t \int_{\mathbf{R}} w_x[f(w+\phi) - f(\phi)]\, dx\, ds \\
&= \frac{1}{d}\int_0^t \int_{\mathbf{R}} \left[\left(\int_{\phi}^{w+\phi} f(y)\, dy - f(\phi)\, w\right)_x \right. \\
&\quad \left. - (f(w+\phi) - f(\phi) - f'(\phi)\, w)\, \phi_x\right] dx\, ds \\
&\leqslant -\frac{\gamma}{2d}\int_0^t \int_{\mathbf{R}} \phi_x w^2\, dx\, ds,
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
I_4 &= -\int_0^t \int_{\mathbf{R}} w_t [f'(w+\phi) - f'(\phi)]\, \phi_x \, dx\, ds - \int_0^t \int_{\mathbf{R}} w_t f'(w+\phi)\, w_x \, dx\, ds \\
&\leqslant \frac{1}{2} \int_0^t \int_{\mathbf{R}} w_t^2 \, dx\, ds + \frac{1}{2} \int_0^t \int_{\mathbf{R}} [f'(w+\phi)]^2 \, w_x^2 \, dx\, ds \\
&\quad - \int_{\mathbf{R}} \phi_x w [f'(w+\phi) - f'(\phi)] \, dx \\
&\quad + \int_{\mathbf{R}} \phi_x w_0 [f'(w_0+\phi) - f'(\phi)] \, dx + \int_0^t \int_{\mathbf{R}} w [f'(w+\phi) - f'(\phi)]\, \phi_{xt} \, dx\, ds \\
&\quad + \int_0^t \int_{\mathbf{R}} w \phi_x \phi_t [f''(w+\phi) - f''(\phi)] \, dx\, ds + \int_0^t \int_{\mathbf{R}} w w_t \phi_x f''(w+\phi) \, dx\, ds \\
&\leqslant O(1)\, \|w_0(x)\|_{L^2(\mathbf{R})}^2 + \frac{1}{2} \int_0^t \int_{\mathbf{R}} w_t^2 \, dx\, ds + \frac{1}{2} \int_0^t \int_{\mathbf{R}} [f'(w+\phi)]^2 \, w_x^2 \, dx\, ds \\
&\quad + O(1) \int_0^t \int_{\mathbf{R}} (|\phi_{xt}| w^2 + |\phi_x \phi_t| w^2 + |w w_t \phi_x|) \, dx\, ds \\
&\leqslant O(1)\, \|w_0(x)\|_{L^2(\mathbf{R})}^2 + \left(\frac{1}{2} + \varepsilon\right) \int_0^t \int_{\mathbf{R}} w_t^2 \, dx\, ds \\
&\quad + \frac{1}{2} \int_0^t \int_{\mathbf{R}} [f'(w+\phi)]^2 \, w_x^2 \, dx\, ds \\
&\quad + O(1) \int_0^t (1+t)^{-3/2} \, \|w(s)\|_{L^2(\mathbf{R})}^2 \, ds. \qquad (2.14)
\end{aligned}
$$

Here we have used the estimate (2.3).

Substituting (2.11)–(2.14) into (2.10), we obtain

$$
\begin{aligned}
&\frac{1}{4d} \int_{\mathbf{R}} w^2 \, dx + \left(\frac{1}{2} - \frac{1}{d}\right) \int_{\mathbf{R}} w_t^2 \, dx + \frac{a_1}{2} \int_{\mathbf{R}} w_x^2 \, dx + \frac{\gamma}{2d} \int_0^t \int_{\mathbf{R}} \phi_x w^2 \, dx\, ds \\
&\quad + \int_0^t \int_{\mathbf{R}} \left(\frac{a_1}{d} - \frac{[f'(w+\phi)]^2}{2}\right) w_x^2 \, dx\, ds + \left(\frac{1}{2} - 2\varepsilon - \frac{1}{d}\right) \int_0^t \int_{\mathbf{R}} w_t^2 \, dx\, ds \\
&\leqslant O(1)(\|w_0(x)\|_{H^1(\mathbf{R})}^2 + \|z_{0x}(x)\|_{L^2(\mathbf{R})}^2 + 1) \\
&\quad + O(1) \int_0^t (1+t)^{-5/4} \|w(s)\|_{L^2(\mathbf{R})}^2 \, ds. \qquad (2.15)
\end{aligned}
$$

From the subcharacteristic condition $(\mathrm{SC})_1$, we can choose a constant $d>2$ and $\varepsilon>0$ sufficiently small such that

$$\begin{cases} \dfrac{a_1}{2}>\varepsilon, \\ \dfrac{1}{2}-2\varepsilon-\dfrac{1}{d}>0. \\ \dfrac{a_1}{d}>\dfrac{[f'(w+\phi)]^2}{2}, \end{cases} \tag{2.16}$$

and consequently

$$\int_{\mathbf{R}} (w^2+w_x^2+w_t^2)\,dx+\int_0^t \int_{\mathbf{R}} (\phi_x w^2+w_x^2+w_t^2)\,dx\,ds$$

$$\leqslant O(1)(\|w_0(x)\|^2_{H^1(\mathbf{R})}+\|z_{0x}(x)\|^2_{L^2(\mathbf{R})}+1)$$

$$+O(1)\int_0^t (1+t)^{-5/4}\,\|w(s)\|^2_{L^2(\mathbf{R})}\,ds. \tag{2.17}$$

Having obtained (2.17), by employing the Gronwall's inequality, we can deduce (2.9) easily. This completes the proof of Lemma 2.4.

For the higher order energy estimates for $w(t,x)$, we have

Lemma 2.5 (Higher Order Energy Estimate). *Under the assumptions of Theorem* 1.1, *we have*

$$\int_{\mathbf{R}} (w_{xx}^2+w_{xt}^2+w_{tt}^2)(t)\,dx+\int_0^t \int_{\mathbf{R}} (w_{xx}^2+w_{xt}^2+w_{tt}^2)(s,x)\,dx\,ds$$

$$\leqslant O(1)(\|w_0(x)\|^2_{H^2(\mathbf{R})}+\|z_{0x}(x)\|^2_{H^1(\mathbf{R})}+1). \tag{2.18}$$

Proof. First, we differentiate $(2.8)_1$ with respect to t and muliply the results by w_{tt}. Second, we multiply $(2.8)_1$ by $-\frac{1}{d}w_{tt}$. Adding the two resulting identities and integrating over $[0,t]\times\mathbf{R}$, after some integrations by parts, we deduce that

$$\frac{1}{2}\int_{\mathbf{R}} w_{tt}^2\,dx+\frac{a_1}{2}\int_{\mathbf{R}} w_{xt}^2\,dx+\frac{a_1}{d}\int_0^t \int_{\mathbf{R}} w_{xt}^2\,dx\,ds$$

$$+\left(1-\frac{1}{d}\right)\int_0^t \int_{\mathbf{R}} w_{tt}^2\,dx\,ds$$

$$
\begin{aligned}
&= O(1)(\|w_0(x)\|^2_{H^2(\mathbf{R})} + \|z_{0x}(x)\|^2_{H^1(\mathbf{R})}) + \frac{1}{2d}\int_{\mathbf{R}} w_t^2\,dx \\
&\quad + \frac{a_1}{d}\int_{\mathbf{R}} w_x w_{xt}\,dx + \frac{1}{d}\int_0^t\int_{\mathbf{R}} w_{tt}[f(w+\phi)-f(\phi)]_x\,dx\,ds \\
&\quad - \int_0^t\int_{\mathbf{R}} w_{tt}[f(w+\phi)-f(\phi)]_{xt}\,dx\,ds \\
&\quad + O(1)\int_0^t\int_{\mathbf{R}} |w_{tt}|\,(|\phi_{xxt}| + |\phi_{xx}| + |f(\phi)_{xt}| + |f(\phi)_{xtt}|)\,dx\,ds \\
&= \sum_{i=6}^{11} I_i. \qquad (2.19)
\end{aligned}
$$

From the Cauchy–Schwarz inequality, Lemma 2.1, (2.9), and (2.3), we get

$$
\begin{aligned}
I_7 + I_8 + I_{11} &\leqslant O(1)(\|w_0\|^2_{H^1(\mathbf{R})} + \|z_{0x}\|^2_{L^2(\mathbf{R})} + 1) \\
&\quad + \varepsilon\int_0^t\int_{\mathbf{R}} w_{tt}^2\,dx\,ds + \varepsilon\int_{\mathbf{R}} w_{xt}^2\,dx, \qquad (2.20)
\end{aligned}
$$

$$
\begin{aligned}
I_9 &\leqslant \frac{\varepsilon}{2}\int_0^t\int_{\mathbf{R}} w_{tt}^2\,dx\,ds + O(1)\int_0^t\int_{\mathbf{R}} (\phi_x w^2 + w_x^2)\,dx\,ds \\
&\leqslant \frac{\varepsilon}{2}\int_0^t\int_{\mathbf{R}} w_{tt}^2\,dx\,ds + O(1)(\|w_0(x)\|^2_{H^1(\mathbf{R})} + \|z_{0x}(x)\|^2_{L^2(\mathbf{R})} + 1), \qquad (2.21)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{10} &= -\int_0^t\int_{\mathbf{R}} f'(w+\phi)\,w_{xt}w_{tt}\,dx\,ds - \int_0^t\int_{\mathbf{R}} [f'(w+\phi)-f'(\phi)]\,\phi_{xt}w_{tt}\,dx\,ds \\
&\quad - \int_0^t\int_{\mathbf{R}} f'(w+\phi)_t\,w_x\,w_{tt}\,dx\,ds - \int_0^t\int_{\mathbf{R}} [f'(w+\phi)-f'(\phi)]_t\,\phi_x w_{tt}\,dx\,ds \\
&\leqslant \frac{1+\varepsilon}{2}\int_0^t\int_{\mathbf{R}} w_{tt}^2\,dx\,ds + \frac{1}{2}\int_0^t\int_{\mathbf{R}} [f'(w+\phi)]^2\,w_{xt}^2\,dx\,ds \\
&\quad + O(1)\int_0^t\int_{\mathbf{R}} (\phi_{xt}^2 w^2 + \phi_x^2(\phi_t^2 w^2 + w_t^2) + w_x^2(w_t^2 + \phi_t^2))\,dx\,ds \\
&\leqslant \frac{1+\varepsilon}{2}\int_0^t\int_{\mathbf{R}} w_{tt}^2\,dx\,ds + \frac{1}{2}\int_0^t\int_{\mathbf{R}} [f'(w+\phi)]^2\,w_{xt}^2\,dx\,ds \\
&\quad + O(1)(\|w_0(x)\|^2_{H^1(\mathbf{R})} + \|z_{0x}(x)\|^2_{L^2(\mathbf{R})} + 1). \qquad (2.22)
\end{aligned}
$$

In the above inequalities, the constants d and ε satisfy (2.16).

Consequently

$$\frac{1}{2}\int_{\mathbf{R}} w_{tt}^2\,dx+\left(\frac{a_1}{2}-\varepsilon\right)\int_{\mathbf{R}} w_{xt}^2\,dx+\int_0^t\int_{\mathbf{R}}\left(\frac{a_1}{d}-\frac{[f'(w+\phi)]^2}{2}\right)w_{xt}^2\,dx\,ds$$
$$+\left(\frac{1}{2}-\frac{1}{d}-2\varepsilon\right)\int_0^t\int_{\mathbf{R}} w_{tt}^2\,dx\,ds$$
$$\leqslant O(1)(\|w_0(x)\|^2_{H^2(\mathbf{R})}+\|z_{0x}(x)\|^2_{H^1(\mathbf{R})}+1),$$

i.e.,

$$\int_{\mathbf{R}}(w_{tt}^2+w_{xt}^2)\,dx+\int_0^t\int_{\mathbf{R}}(w_{xt}^2+w_{tt}^2)\,dx\,ds\leqslant O(1)(\|w_0\|^2_{H^2(\mathbf{R})}+\|z_{0x}\|^2_{H^1(\mathbf{R})}+1). \tag{2.23}$$

With (2.23) in hand and noticing

$$a_1 w_{xx}=w_{tt}+w_t+[f(w+\phi)-f(\phi)]_x-(a_1\phi_{xx}+f(\phi)_{xt}),$$

we can easily deduce that

$$\int_{\mathbf{R}} w_{xx}^2\,dx+\int_0^t\int_{\mathbf{R}} w_{xx}^2\,dx\,ds\leqslant O(1)(\|w_0(x)\|^2_{H^2(\mathbf{R})}+\|z_{0x}(x)\|^2_{H^1(\mathbf{R})}+1), \tag{2.24}$$

and (2.18) follows from (2.23) and (2.24). This completes the proof of Lemma 2.5.

Lemma 2.3 follows from Lemma 2.4 and Lemma 2.5 and thus the proof of Theorem 1.1 is complete.

3. THE PROOF OF THEOREM 1.2

This section is devoted to proving Theorem 1.2. To this end, we use $(\bar{u}(t,x),\bar{v}_1(t,x))$ to denote the unique global smooth solution to the system $(1.11)_1$ and $(1.11)_2$ with initial data $(\bar{u}(0,x),\bar{v}_1(0,x))=(\phi_0(x),f(\phi_0(x)))$, and let $(\bar{w}(t,x),\bar{z}(t,x))=(\bar{u}(t,x)-\phi(t,x),\bar{v}_1(t,x)-f(\phi(t,x)))$, then we can deduce that $(\bar{u}(t,x),\bar{v}_1(t,x))$ and $\bar{w}(t,x)$ have the following properties

LEMMA 3.1. *Under the above notations, we have that*

(i) $\bar{u}_x(t,x)\geqslant 0$, $|\bar{u}_t(t,x)|\leqslant\sqrt{a_1}\,\bar{u}_x(t,x)$, $t\geqslant 0$, $x\in\mathbf{R}$;

(ii) $\|\partial^i\bar{u}(t,x)/\partial x^i\|_{L^\infty(\mathbf{R})}\leqslant O(1)$, $i=1,2,\cdots,7$;

(iii) $\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\{|\bar{w}_x(t,x)|\}=0$;

(iv) $\int_0^\infty\int_{\mathbf{R}}(|\bar{w}_x(t,x)|^2+|\bar{w}_{xx}(t,x)|^2)\,dx\,dt\leqslant O(1)$.

Proof. (i) and (ii) follow from Lemma 2.1 and Lemma 2.3 in [15] respectively.

As to (iii) and (iv), notice that $(\bar{w}(t,x),\bar{z}(t,x))$ solves the Cauchy problem

$$\begin{cases}\bar{w}_t+\bar{z}_x=0,\\ \bar{z}_t+a_1\bar{w}_x+\bar{z}=[f(\bar{w}+\phi)-f(\phi)]-(a_1\phi_x+f(\phi)_t),\\ (\bar{w}(0,x),\bar{z}(0,x))=(\bar{w}_0(x),\bar{z}_0(x))=(0,0),\end{cases}\tag{3.1}$$

thus Theorem 1.1 and Lemma 2.3 can be applied directly to this case to deduce that (iii) and (iv) hold. This completes the proof of Lemma 3.1.

Combining Lemma 2.1 with Lemma 3.1, we have

COROLLARY 3.2. *$\bar{u}(t,x)$ satisfies*

(i) $\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\{|\bar{u}_t(t,x)|\}=0$;

(ii) $\int_0^\infty(\int_{\mathbf{R}}|\bar{u}_x(t,x)|^4\,dx)^{2/3}\,dt\leqslant O(1)$.

Proof. Due to

$$|\bar{u}_t(t,x)|\leqslant\sqrt{a_1}\,\bar{u}_x(t,x)\leqslant\sqrt{a_1}\,(|\bar{w}_x(t,x)|+|\phi_x(t,x)|),$$

we can easily get (i) from (iii) of Lemma 3.1 and $(2.1)_1$.

Now we turn to prove (ii) of Corollary 3.2.

Notice that

$$\bar{u}(t,x)=\bar{w}(t,x)+\phi(t,x),\tag{3.2}$$

we have

$$\begin{aligned}\int_{\mathbf{R}}|\bar{u}_x(t,x)|^4\,dx&\leqslant O(1)\left(\int_{\mathbf{R}}|\bar{w}_x(t,x)|^4\,dx+\int_{\mathbf{R}}|\phi_x(t,x)|^4\,dx\right)\\&\leqslant O(1)((1+t)^{-3}+\|\bar{w}_x(t,x)\|^2_{L^\infty(\mathbf{R})}\|\bar{w}_x(t,x)\|^2_{L^2(\mathbf{R})})\\&\leqslant O(1)((1+t)^{-3}+\|\bar{w}_{xx}(t,x)\|_{L^2(\mathbf{R})}\|\bar{w}_x(t,x)\|^3_{L^2(\mathbf{R})})\\&\leqslant O(1)((1+t)^{-3}+\|\bar{w}_{xx}(t,x)\|^4_{L^2(\mathbf{R})}+\|\bar{w}_x(t,x)\|^4_{L^2(\mathbf{R})}).\end{aligned}$$

Consequently, from (iv) of Lemma 3.1, we get

$$\int_0^\infty \left(\int_{\mathbf{R}} |\bar{u}_x(t,x)|^4 \, dx \right)^{2/3} dt \leqslant O(1) \left(\int_0^\infty (1+t)^{-2} \, dt \right.$$
$$\left. + \int_0^\infty \int_{\mathbf{R}} \left(|\bar{w}_{xx}(t,x)|^2 + |\bar{w}_x(t,x)|^2 \right) dx \, dt \right)$$
$$\leqslant O(1). \tag{3.3}$$

This completes the proof of Corollary 3.2.

From Theorem 1.1, to prove Theorem 1.2, we only need to compare the solution $(u, v_1, v_2)(t, x, y)$ of (1.4)–(1.5) with the functions $(\bar{u}(t,x), \bar{v}_1(t,x), g(\bar{u}(t,x)))$. For this purpose, let

$$\begin{cases} U(t,x,y) = u(t,x,y) - \bar{u}(t,x), \\ V_1(t,x,y) = v_1(t,x,y) - \bar{v}_1(t,x), \\ V_2(t,x,y) = v_2(t,x,y) - g(\bar{u}(t,x)), \end{cases} \tag{3.4}$$

then, $(U, V_1, V_2)(t, x, y)$ solves the following Cauchy problem

$$\begin{cases} U_t + V_{1x} + V_{2y} = 0, \\ V_{1t} + a_1 U_x = f(\bar{u} + U) - f(\bar{u}) - V_1, \\ V_{2t} + a_2 U_y = g(\bar{u} + U) - g(\bar{u}) - V_2 - g(\bar{u})_t, \end{cases} \tag{3.5}$$

with initial data

$$\begin{cases} U(0,x,y) = U_0(x,y) = u_0(x,y) - \phi_0(x), \\ V_1(0,x,y) = V_{10}(x,y) = v_{10}(x,y) - f(\phi_0(x)), \\ V_2(0,x,y) = V_{20}(x,y) = v_{20}(x,y) - g(\phi_0(x)). \end{cases} \tag{3.6}$$

It is easy to obtain from (3.5) and (3.6) that [15]

$$V_1(t,x,y) = e^{-t} V_{10}(x,y) + \int_0^t e^{s-t} [f(\bar{u}+U) - f(\bar{u}) - a_1 U_x](s,x,y) \, ds, \tag{3.7}$$

$$V_2(t,x,y) = e^{-t} V_{20}(x,y)$$
$$+ \int_0^t e^{s-t} [g(\bar{u}+U) - g(\bar{u}) - a_2 U_y - g(\bar{u})_t](s,x,y) \, ds. \tag{3.8}$$

Substituting (3.7) and (3.8) into $(3.5)_1$, we can deduce that $U(t, x, y)$ solves the Cauchy problem

$$U_{tt}+[f(\bar{u}+U)-f(\bar{u})]_x-a_1 U_{xx}+g(\bar{u}+U)_y-a_2 U_{yy}+U_t=0, \tag{3.9}$$

with initial data

$$\begin{cases} U(0, x, y)=U_0(x, y)\in H^4(\mathbf{R}^2), \\ U_t(0, x, y)=-V_{10x}(x, y)-V_{20y}(x, y)\in H^3(\mathbf{R}^2). \end{cases} \tag{3.10}$$

Following the arguments developed by T. Luo in [15], to prove Theorem 1.2, we first consider the Cauchy problem (3.9)–(3.10) in the Banach space

$$\begin{aligned} X(0, T)=\{&U(t, x, y) \mid U(t, x, y) \\ &\in C^0(0, T; H^4(\mathbf{R}^2)),\ U_t(t, x, y)\in L^2(0, T; H^3(\mathbf{R}^2)), \\ &U_x(t, x, y), U_y(t, x, y)\in L^2(0, T; H^3(\mathbf{R}^2))\}. \end{aligned}$$

We assume that the solution $U(t, x, y)$ of (3.9)–(3.10) belongs to $X(0, T)$ for some $T>0$ (by the classical local existence result, see, for example, A. Majda [17], such a T does exist) and let

$$N(t)=\sup_{s\in[0,t]}\left(\|U(s, x, y)\|_{H^4(\mathbf{R}^2)}+\|U_t(s, x, y)\|_{H^3(\mathbf{R}^2)}\right).$$

In what follows, we always assume that $N(T)\leqslant \varepsilon_0/10$ for each given constant $\varepsilon_0>0$. Consequently

$$\begin{cases} \sup\limits_{t\in[0,T]}\left(\|U(t, x, y)\|_{C^2(\mathbf{R}^2)}+\|U_t(t, x, y)\|_{C^1(\mathbf{R}^2)}\right)\leqslant 10N(T)\leqslant\varepsilon_0, \\ \sup\limits_{(t,y)\in[0,T]\times\mathbf{R}}\left\{\left\|\dfrac{\partial^{i+j}U(t, x, y)}{\partial x^i\partial y^j}\right\|_{L^2(\mathbf{R}^2{}_x)}\right\}\leqslant N(T)^2,\ i\geqslant 0,\ j\geqslant 0,\ i+j\leqslant 3, \\ \sup\limits_{(t,x)\in[0,T]\times\mathbf{R}}\left\{\left\|\dfrac{\partial^{i+j}U(t, x, y)}{\partial x^i\partial y^j}\right\|_{L^2(\mathbf{R}^2{}_y)}\right\}\leqslant N(T)^2,\ i\geqslant 0,\ j\geqslant 0,\ i+j\leqslant 3. \end{cases} \tag{3.11}$$

Now we are ready to obtain the following basic energy estimates to the solution $U(t, x, y)$ of the Cauchy problem (3.9)–(3.10).

LEMMA 3.3 (Basic Energy Estimate). *For each given positive constant* $\varepsilon_0 > 0$, *if* $N(T) \leqslant \varepsilon_0/10$, *then the solution* $U(t, x, y)$ *of the Cauchy problem* (3.9)–(3.10) *satisfies the following basic energy estimate*: *For each* $0 < t \leqslant T$,

$$\|U(t)\|_1^2 + \|U_t(t)\|^2 + \int_0^t (\|U_t(s)\|^2 + \|U_x(s)\|^2 + \|U_y(s)\|^2)\, ds$$
$$+ \iiint \bar{u}_x(s, x)\, U^2(s, x, y)\, dx\, dy\, ds \leqslant C_1(\varepsilon_0)(\|U_0\|_1^2 + \|V_{10x}\|^2 + \|V_{20y}\|^2). \tag{3.12}$$

Proof. For each given $\varepsilon_0 > 0$, under the assumptions of Lemma 3.3, we have that $N(T) \leqslant \varepsilon_0/10$. Notice also that $|\bar{u}(t, x)| \leqslant B(N_1)$, we conclude that for each $t \in [0, T]$, $u(t, x, y) = \bar{u}(t, x) + U(t, x, y) \in M$ and thus from the subcharacteristic condition $(\mathrm{SC})_2$, if we let

$$\begin{cases} k_1 = \sup\limits_{u \in M} \{[f'(u)]^2\}, \\ k_2 = \sup\limits_{u \in M} \{[g'(u)]^2\}, \end{cases} \tag{3.13}$$

then we have

$$k = \frac{k_1}{a_1} + \frac{k_2}{a_2} \in (0, 1). \tag{3.14}$$

Consequently we can choose

$$\lambda = \frac{3-k}{2-k}, \qquad \lambda_1 \in (1, \lambda), \tag{3.15}$$

and $\varepsilon > 0$ sufficiently small such that

$$\begin{cases} a_1 - \dfrac{\lambda}{2} a_1 - \varepsilon > 0, \\ a_2 - \dfrac{\lambda}{2} a_2 > 0, \\ \lambda - 1 - \dfrac{1}{2} \lambda - \varepsilon > 0,\ 1 < \lambda < 2. \end{cases} \tag{3.16}$$

With the constants λ, λ_1, and ε chosen as above, multiplying (3.9) by U and λU_t respectively, adding the results, and integrating the resulting equation over $[0, t] \times \mathbf{R}^2$ gives

$$\frac{1}{2}\|U(t)\|^2+\frac{\lambda}{2}\{\|U_t(t)\|^2+a_1\|U_x(t)\|^2+a_2\|U_y(t)\|^2\}$$

$$+\iiint\{a_1U_x^2+a_2U_y^2+(\lambda-1)\,U_t^2\}(s,x,y)\,dx\,dy\,ds$$

$$+\iint(UU_t)(t,x,y)\,dx\,dy$$

$$+\iiint\{[f(U+\bar{u})-f(\bar{u})]_x\,U+g(U+\bar{u})_y\,U\}(s,x,y)\,dx\,dy\,ds$$

$$+\lambda\iiint\{[f(U+\bar{u})-f(\bar{u})]_x\,U_t+g(U+\bar{u})_y\,U_t\}(s,x,y)\,dx\,dy\,ds$$

$$=\iint(UU_t)(0,x,y)\,dx\,dy+\frac{1}{2}\|U_0\|^2$$

$$+\frac{\lambda}{2}\{\|U_t(0,x,y)\|^2+a_1\|U_{0x}\|^2+a_2\|U_{0y}\|^2\}. \tag{3.17}$$

Each term in (3.17) will be estimated seperately as follows. First, following exactly the arguments used by T. Luo in [15], we can get

$$\left|\lambda\iiint\{g(U+\bar{u})_y\,U_t\}(s,x,y)\,dx\,dy\,ds\right|\leqslant\frac{\lambda a_2}{2}\iiint U_y^2(s,x,y)\,dx\,dy\,ds$$

$$+\frac{\lambda k_2}{2a_2}\iiint U_t^2(s,x,y)\,dx\,dy\,ds, \tag{3.18}$$

and

$$\left|\iint(UU_t)(t,x,y)\,dx\,dy\right|\leqslant\frac{1}{2\lambda_1}\iint U^2(t,x,y)\,dx\,dy+\frac{\lambda_1}{2}\iint U_t^2(t,x,y)\,dx\,dy. \tag{3.19}$$

Second, using intergrations by parts leads to

$$\iiint\{[f(U+\bar{u})-f(\bar{u})]_x\,U\}(s,x,y)\,dx\,dy\,ds$$

$$=-\iiint\{[f(U+\bar{u})-f(\bar{u})]\,U_x\}(s,x,y)\,dx\,dy\,ds$$

$$= \iiint \Big\{ -\Big[\int_{\bar u}^{U+\bar u} f(z)\, dz - f(\bar u)\, U \Big]_x$$

$$+ [f(U+\bar u) - f(\bar u) - f'(\bar u)\, U]\, \bar u_x \Big\}(s, x, y)\, dx\, dy\, ds$$

$$\geqslant \frac{\gamma}{2} \iiint \bar u_x(s, x)\, U^2(s, x, y)\, dx\, dy\, ds. \tag{3.20}$$

Third, we treat the term $\lambda \iiint \{[f(U+\bar u) - f(\bar u)]_x U_t\}(s, x, y)\, dx\, dy\, ds$, which is the main novelty of this paper.

Note that

$$\lambda \iiint \{[f(U+\bar u) - f(\bar u)]_x\, U_t\}(s, x, y)\, dx\, dy\, ds$$

$$= \lambda \iiint \{[f'(U+\bar u) - f'(\bar u)]\, \bar u_x U_t + f'(U+\bar u)\, U_x U_t\}(s, x, y)\, dx\, dy\, ds. \tag{3.21}$$

It holds from the Cauchy–Schwarz inequality that

$$\left|\lambda \iiint \{f'(U+\bar u)\, U_x U_t\}(s, x, y)\, dx\, dy\, ds\right| \leqslant \frac{\lambda a_1}{2} \iiint U_x^2(s, x, y)\, dx\, dy\, ds$$

$$+ \frac{\lambda k_1}{2a_1} \iiint U_t^2(s, x, y)\, dx\, dy\, ds, \tag{3.22}$$

On the other hand

$$\left|\lambda \iiint \{[f'(U+\bar u) - f'(\bar u)]\, \bar u_x U_t\}(s, x, y)\, dx\, dy\, ds\right|$$

$$\leqslant O(1) \iiint (\bar u_x |U|\, |U_t|)(s, x, y)\, dx\, dy\, ds$$

$$\leqslant \varepsilon \iiint U_t^2(s, x, y)\, dx\, dy\, ds + O(1) \iiint (\bar u_x^2 U^2)(s, x, y)\, dx\, dy\, ds. \tag{3.23}$$

Since

$$\int_{\mathbf R} (\bar u_x^2 U^2)(s, x, y)\, dx \leqslant \Big(\int_{\mathbf R} \bar u_x^4(s, x)\, dx \Big)^{1/2} \Big(\int_{\mathbf R} U^4(s, x, y)\, dx \Big)^{1/2}$$

$$\leqslant \Big(\int_{\mathbf R} \bar u_x^4(s, x)\, dx \Big)^{1/2} \| U(s, x, y)\|_{L^2(\mathbf R_x)}^{3/2}\, \| U_x(s, x, y)\|_{L^2(\mathbf R_x)}^{1/2}$$

$$\leqslant O(1)\, \varepsilon \int_{\mathbf R} U_x^2(s, x, y)\, dx + O(1) \Big(\int_{\mathbf R} \bar u_x^4(s, x)\, dx \Big)^{2/3} \int_{\mathbf R} U^2(s, x, y)\, dx,$$

we have

$$\left|\lambda \iiint \{[f'(U+\bar{u})-f'(\bar{u})]\,\bar{u}_x U_t\}(s,x,y)\,dx\,dy\,ds\right|$$
$$\leqslant \varepsilon \iiint (U_t^2+U_x^2)(s,x,y)\,dx\,dy\,ds$$
$$+O(1)\int_0^t \left(\int_{\mathbf{R}} \bar{u}_x^4(s,x)\,dx\right)^{2/3} \|U(s)\|^2\,ds. \tag{3.24}$$

Substituting (3.18), (3.19), (3.20), (3.22), and (3.24) into (3.17) and noticing that

$$\iiint (Ug(U+\bar{u})_y)(s,x,y)\,dx\,dy\,ds = -\iiint (U_y g(U+\bar{u}))(s,x,y)\,dx\,dy\,ds$$
$$= -\iiint \left[\int_{\bar{u}}^{U+\bar{u}} g(z)\,dz\right]_y (s,x,y)\,dx\,dy\,ds$$
$$=0, \tag{3.25}$$

we conclude that

$$\frac{1}{2}\left(1-\frac{1}{\lambda_1}\right)\|U(t)\|^2+\frac{\lambda}{2}\left\{\left(1-\frac{\lambda}{\lambda_1}\right)\|U_t(t)\|^2+a_1\|U_x(t)\|^2+a_2\|U_y(t)\|^2\right\}$$
$$+\iiint\left\{\left[a_1-\frac{\lambda a_1}{2}-\varepsilon\right]U_x^2+\left[a_2-\frac{\lambda a_2}{2}\right]U_y^2\right.$$
$$\left.+\left[\lambda-1-\frac{\lambda k}{2}-\varepsilon\right]U_t^2\right\}(s,x,y)\,dx\,dy\,ds$$
$$+\frac{\gamma}{2}\iiint(\bar{u}_x U^2)(s,x,y)\,dx\,dy\,ds \leqslant O(1)(\|U_0\|_1^2+\|V_{10x}\|^2+\|V_{20y}\|^2)$$
$$+O(1)\int_0^t\left(\int_{\mathbf{R}}\bar{u}_x^4(s,x)\,dx\right)^{2/3}\|U(s)\|^2\,ds. \tag{3.26}$$

Since ε, λ, λ_1 satisfy (3.16), we have from (3.26) that

$$\|U(t)\|_1^2+\|U_t(t)\|^2+\int_0^t(\|U_x(s)\|^2+\|U_t(s)\|^2+\|U_y(s)\|^2)\,ds$$
$$+\iiint(\bar{u}_x U^2)(s,x,y)\,dx\,dy\,ds \leqslant O(1)(\|U_0\|_1^2+\|V_{10x}\|^2+\|V_{20y}\|^2)$$
$$+O(1)\int_0^t\left(\int_{\mathbf{R}}\bar{u}_x^4(s,x)\,dx\right)^{2/3}\|U(s)\|^2\,ds. \tag{3.27}$$

Having obtained (3.27), Gronwall's inequality and (ii) of Corollary 3.2 implies that

$$\|U(t)\|^2 \leqslant O(1)(\|U_0\|_1^2 + \|V_{10x}\|^2 + \|V_{20y}\|^2), \tag{3.28}$$

and (3.12) follows from (3.27), (3.28), and (ii) of Corollary 3.2. This completes the proof of Lemma 3.3.

As to the energy estimates on higher order derivatives of $U(t, x, y)$, if we check the arguments used by T. Luo in [15] carefully, we can find that what his arguments based on is that the *a priori* hypothesis $N(T) \leqslant \varepsilon_0/10$, the subcharacteristic condition $(\mathrm{SC})_2$, the basic energy estimate (3.12), and the fact that $\bar{u}(t, x)$ satisfies (ii) of Lemma 3.1, but the smallness conditions imposed on ε_0 and δ are unnecessary. From this observation, we have the following results on higher order energy estimates

LEMMA 3.4 (Higher Order Energy Estimates). *Under the conditions of Lemma* 3.3, *we have for each* $0 \leqslant t \leqslant T$

$$\|U(t)\|_4^2 + \|U_t(t)\|_3^2 + \int_0^t \|(U_x, U_y, U_t)(s)\|_3^2 \, ds \leqslant C_2(\varepsilon_0) \, N(0)^2. \tag{3.29}$$

With Lemma 3.4 in hand, we now consider the global solvability of the Cauchy problem (3.9)–(3.10).

THEOREM 3.5. *Suppose that* $U_0(x, y) \in H^4(\mathbf{R}^2)$, $U_t(0, x, y) \in H^3(\mathbf{R}^2)$, *then for each given* $\varepsilon_0 > 0$, *there exists a constant* $\alpha_0 > 0$ *such that if* $\alpha \leqslant \alpha_0$, *the Cauchy problem* (3.9)–(3.10) *admits a unique global smooth solution* $U(t, x, y)$ *which satisfies* (3.29) *and*

$$\|U(t, x, y)\|_{L^\infty(\mathbf{R}^2)} \leqslant \varepsilon_0. \tag{3.30}$$

Proof. From the classical local existence results (see, for example, A. Majda [17]), the Cauchy problem (3.9)–(3.10) has a unique smooth solution $U(t, x, y)$ on $\prod_{t_1} = \{(t, x, y) \mid 0 \leqslant t \leqslant t_1, (x, y) \in \mathbf{R}^2\}$ where t_1 depends only on $\|U_0\|_4$ and $\|U_t(0, x, y)\|_3$. Suppose that $U(t, x, y)$ has been extended to the time $t = T > t_1$, from Lemma 3.4, we know that if $U(t, x, y)$ satisfies the *a priori* hypothesis

$$N(T) \leqslant \frac{\varepsilon_0}{10}, \tag{3.31}$$

then we have

$$N(t)^2 \leqslant C_2(\varepsilon_0) \, \alpha^2, \qquad 0 \leqslant t \leqslant T. \tag{3.32}$$

Now from the standard continuity argument, to show $T = +\infty$, we only need to show that the *a priori* hypothesis (3.31) can be closed on $t \in [0, T]$. This is the case if we choose $\alpha_0 > 0$ such that

$$\alpha_0 \leqslant \frac{\varepsilon_0}{10\sqrt{C_2(\varepsilon_0)}}, \tag{3.32}$$

and let the initial data to satisfy $\alpha < \alpha_0$. This completes the proof of Theorem 3.5.

Now we turn to prove Theorem 1.2. We first consider the global existence results.

Notice that if $(u, v_1, v_2)(t, x, y)$ is a smooth solution to the Cauchy problem (1.4)–(1.5) on the strip $\prod_T$, then $(U, V_1, V_2)(t, x, y)$ solves the Cauchy problem (3.5)–(3.6) on $\prod_T$ and consequently $U(t, x, y)$ solves the Cauchy problem (3.9)–(3.10) on $\prod_T$. But from Theorem 3.5, under the conditions of Theorem 1.2, the smooth solution $U(t, x, y)$ to the Cauchy problem (3.9)–(3.10) on $\prod_T$ can be extended to $T = +\infty$. If we define $V_1(t, x, y)$, $V_2(t, x, y)$ by using $U(t, x, y)$, $\bar{u}(t, x)$ as in (3.7) and (3.8), respectively, we can easily deduce that $(U, V_1, V_2)(t, x, y)$ solves (3.5)–(3.6) globally and the corresponding $(u, v_1, v_2)(t, x, y)$ solves (1.4)–(1.5) globally. Since $(u, v_1, v_2)(t, x, y)$ obtained above is a global smooth solution to the Cauchy problem (1.4)–(1.5), the uniqueness resulits follows easily. This completes the proof of the global existence results of Theorem 1.2.

Finally, we prove the temporal decay estimate (1.19). To this end, from Theorem 1.1, we only need to prove

$$\lim_{t \to +\infty} \|(U, V_1, V_2)(t, x, y)\|_{L^\infty(\mathbf{R}^2)} = 0. \tag{3.33}$$

First we show that

$$\lim_{t \to +\infty} \|U(t, x, y)\|_{L^\infty(\mathbf{R}^2)} = 0. \tag{3.34}$$

To prove (3.34), we get from (3.29) that

$$\int_0^\infty \left\{ \|(U_x, U_y)(t)\|^2 + \left| \frac{d}{dt} \|(U_x, U_y)(t)\|^2 \right| \right\} dt < +\infty, \tag{3.35}$$

from which it follows that

$$\lim_{t \to +\infty} \|(U_x, U_y)(t)\| = 0. \tag{3.36}$$

On the other hand, due to

$$\|U(t)\|_{L^\infty(\mathbf{R}^2)} \leqslant 2\,\|U(t)\|\,\|U_y(t)\| + 2\,\|U_{xy}(t)\|\,\|U_x(t)\|, \tag{3.37}$$

we can immediately obtain (3.34) from (3.36) and (3.37).

Similarly, we can get

$$\lim_{t\to+\infty} \|(U_x, U_y)(t)\|_{L^\infty(\mathbf{R}^2)} = 0. \tag{3.38}$$

Thus it holds from (3.7), (3.34), and (3.38) that

$$\lim_{t\to+\infty} \|V_1(t, x, y)\|_{L^\infty(\mathbf{R}^2)} = 0. \tag{3.39}$$

Noticing (i) of Corollary 3.2, the corresponding estimate on $V_2(t, x, y)$ can be proved similarly from (3.8), (3.34), and (3.38). This completes the proof of Theorem 1.2.

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